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Commutativity of the adjacency matrices of graphs

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ABSTRACT

We say that two graphs G_1 and G_2 with the same vertex set commute if their adjacency matrices commute. In this paper, we find all integers n such that the complete bipartite graph $K_{n,n}$ is decomposable into commuting perfect matchings or commuting Hamilton cycles. We show that there are at most $n-1$ linearly independent commuting adjacency matrices of size n; and if this bound occurs, then there exists a Hadamard matrix of order n. Finally, we determine the centralizers of some families of graphs.

1. Introduction

For a graph G with the vertex set $\{v_1, \dots, v_n\}$, the *adjacency matrix* of G is the matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ in which $a_{ij} = 1$, if v_i and v_j are adjacent; and otherwise, $a_{ij} = 0$. We say that two graphs G_1 and G_2 with the same vertex set *commute* if the adjacency matrices of G_1 and G_2 commute. A *graphical matrix* is a symmetric $(0, 1)$-matrix all of whose diagonal entries are 0. If A is a graphical matrix, then the graph corresponding to A denoted by $G(A)$. The *centralizer* of a graph G, denoted $\mathcal{C}(G)$, is the set of all graphical matrices which commute with the adjacency matrix of G. The graph G is called *self-centralizer* if $G \simeq G(A)$, for any non-zero matrix $A \in \mathcal{C}(G)$. For any $n \geqslant 2$, we use the symbols J_n and $\mathbf{j}_n$ for the $n \times n$ matrix and the $1 \times n$ matrix all of whose entries are 1, respectively. Also the identity matrix of size n will be denoted by I_n. For any i, j, $1 \leqslant i, j \leqslant n$, we denote by E_{ij}, that element in $M_n(\mathbb{R})$ whose (i, j)-entry is 1 and whose other entries are 0. A *Hadamard matrix* of order n is a $(-1, 1)$-matrix $H \in M_n(\mathbb{R})$ such that $HH^{\mathsf{T}} = nI_n$.

For a vertex v of a graph G, the *degree* of v is the number of edges incident with v. A graph G is called *k-regular* if the degree of each vertex of G is k. A *perfect matching* M in a graph G is a 1-regular subgraph of G that contains all vertices of G. We denote the path and the cycle with n vertices by P_n and C_n, respectively. A cycle in G that contains all vertices of G is called a *Hamilton cycle*. We denote the complete graph with n vertices and the complete bipartite graph with two parts of sizes n and m, by K_n and $K_{n,m}$, respectively.

In [1], it has been proven that for any $n \geqslant 2$, the complete graph K_n is decomposable into commuting perfect matchings if and only if n is a power of 2. Also, it has been shown that K_n is decomposable into commuting Hamilton cycles if and only if n is a prime. In this paper, we investigate the decompositions of the complete bipartite graphs $K_{n,n}$ into commuting perfect matchings and commuting Hamilton cycles. Also, we determine the centralizers of some families of graphs.

2. Commuting decompositions of $K_{n,n}$

In the first theorem of this section, we wish to determine all integers n for which the graph $K_{n,n}$ is decomposable into commuting perfect matchings.

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Theorem 1. *Let n be a natural number. The graph $K_{n,n}$ is decomposable into commuting perfect matchings if and only if n is a power of 2.*

Proof. First assume that n is a power of 2. By Theorem 1 of [1], we can decompose K_n into $n-1$ commuting perfect matchings with adjacency matrices $B_1, \ldots, B_{n-1}$. Hence $J_n - I_n = B_1 + \cdots + B_{n-1}$. If A is the adjacency matrix of $K_{n,n}$, then using an appropriate labeling of the vertices of $K_{n,n}$, we may suppose that

$$A = \begin{bmatrix} 0 & J_n \\ J_n & 0 \end{bmatrix} = A_0 + \cdots + A_{n-1},$$

where

$$A_0 = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \quad \text{and} \quad A_i = \begin{bmatrix} 0 & B_i \\ B_i & 0 \end{bmatrix}, \quad \text{for } i = 1, \ldots, n-1.$$

Clearly, for any $i = 0, \ldots, n-1$, A_i is the adjacency matrix of a perfect matching of $K_{n,n}$. Moreover, since $\{B_1, \ldots, B_{n-1}\}$ is a commuting set, $\{A_0, \ldots, A_n\}$ is too. Therefore $K_{n,n}$ is decomposable into commuting perfect matchings. For the other direction, suppose that $A_1, \ldots, A_n$ are the adjacency matrices related to a decomposition of $K_{n,n}$ into commuting prefect matchings. With an appropriate labeling of the vertices of $K_{n,n}$, we have

$$A_i = \begin{bmatrix} 0 & B_i \\ B_i^{\mathrm{T}} & 0 \end{bmatrix} \quad \text{for } i = 1, \ldots, n,$$

where $B_1 = I_n$. The equality $A_1 A_i = A_i A_1$ implies that $B_i = B_i^{\mathrm{T}}$. Moreover, for any i, j the equality $A_i A_j = A_j A_i$ holds, so we deduce that $B_i B_j = B_j B_i$. Also, since $B_1 = I_n$ and

$$\begin{bmatrix} 0 & J_n \\ J_n & 0 \end{bmatrix} = \begin{bmatrix} 0 & B_1 \\ B_1 & 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 & B_n \\ B_n & 0 \end{bmatrix},$$

we conclude that $J_n - I_n = B_2 + \cdots + B_n$. Since for $i = 2, \ldots, n$, $G(A_i)$ is a perfect matching of $K_{n,n}$, the latter equation shows that B_i is a permutation $(0, 1)$-matrix all of whose diagonal entries are 0. Hence $G(B_2), \ldots, G(B_n)$ are commuting perfect matchings of K_n that decompose K_n. Now, by Theorem 1 of [1], n is a power of 2. □

Lemma 2. *Let n be an odd number. If B is the adjacency matrix of the cycle C_n, then*

$$A = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$$

is the adjacency matrix of a Hamilton cycle in $K_{n,n}$.

Proof. Clearly, A is a graphical matrix and $G(A)$ is a 2-regular graph. Using Theorem 3.1.2 of [4], we have

$$\det(xI_{2n} - A) = \det \begin{bmatrix} xI_n & -B \\ -B & xI_n \end{bmatrix} = \det(xI_n - B)\det(xI_n + B). \tag{1}$$

Since C_n is a connected 2-regular graph, 2 is a simple eigenvalue of B. Furthermore, since n is odd, C_n is not a bipartite graph and so by Theorem 31.9 of [5], -2 is not an eigenvalue of B. Hence (1) shows that 2 is a simple eigenvalue of A. Since $G(A)$ is a 2-regular, it is a Hamilton cycle in $K_{n,n}$, as desired. □

Let $\mathcal{D} = \{G_0, \ldots, G_k\}$ be a decomposition of a graph G into some of its subgraphs. We say that $\mathcal{D}$ is *quasi-Hamiltonian* if G_0 is a perfect matching in G and $G_1, \ldots, G_k$ are Hamilton cycles of G.

Theorem 3. *Let n be an odd number. The graph $K_{n,n}$ has a commuting quasi-Hamiltonian decomposition if and only if n is a prime number.*

Proof. Let $n = 2s + 1$, for some $s \geqslant 0$. First suppose that n is a prime number and

$$A = \begin{bmatrix} 0 & J_n \\ J_n & 0 \end{bmatrix}$$

is the adjacency matrix of $K_{n,n}$. From Corollary 2 of [1], we know that K_n is decomposable into commuting Hamilton cycles with adjacency matrices $B_1, \ldots, B_s$. Since $J_n - I_n = B_1 + \cdots + B_s$, using Lemma 2, we conclude that

$$\mathcal{P} = \left\{ G\left(\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}\right), G\left(\begin{bmatrix} 0 & B_1 \\ B_1 & 0 \end{bmatrix}\right), \ldots, G\left(\begin{bmatrix} 0 & B_s \\ B_s & 0 \end{bmatrix}\right) \right\}$$

is a quasi-Hamiltonian decomposition of $K_{n,n}$. Moreover, since $\{B_1, \dots, B_s\}$ is a commuting set, it is easily verified that every pair of graphs in $\mathcal{P}$ commutes, as desired. For the other direction, assume that $\{A_0, \dots, A_s\}$ is a commuting quasi-Hamiltonian decomposition of $K_{n,n}$. By choosing an appropriate labeling of the vertices of $K_{n,n}$, we may suppose that

$$A_0 = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \quad \text{and} \quad A_i = \begin{bmatrix} 0 & B_i \\ B_i^{\mathsf{T}} & 0 \end{bmatrix}, \quad \text{for } i = 1, \dots, s.$$

Since $A_0A_i = A_iA_0$, $B_i = B_i^{\mathsf{T}}$, for any $i \geqslant 1$. Moreover, $J_n - I_n = B_1 + \cdots + B_s$, hence B_i is a graphical matrix and obviously $G(B_i)$ is a 2-regular graph, for $i = 1, \dots, s$. Since $G(A_i)$ is a 2-regular connected graph, 2 is a simple eigenvalue of A_i. By applying (1), we conclude that the multiplicity of eigenvalue 2 of B_i is 1, for $i = 1, \dots, s$. This implies that $G(B_i)$ is a connected graph and therefore $\mathcal{Q} = \{G(B_1), \dots, G(B_s)\}$ is a decomposition of K_n into Hamilton cycles. Furthermore, $\{A_1, \dots, A_s\}$ is a commuting set, so it can be easily checked that any pair of graphs in $\mathcal{Q}$ commutes. Now, Corollary 4 of [1] yields that n is a prime number. $\square$

Theorem 4. *Let n be an even number. The graph $K_{n,n}$ is decomposable into commuting Hamilton cycles if and only if n is a power of 2.*

Proof. For each permutation π on $\{1, \dots, m\}$, assume that $P_\pi = [p_{ij}] \in M_m(\mathbb{R})$ is the permutation matrix corresponding to π, that is, $p_{ij} = 1$ if and only if $j = \pi(i)$. Clearly, $G(P_\pi + P_\pi^{\mathsf{T}})$ is a Hamilton cycle in K_m if and only if π is a cycle of length m. First suppose that $K_{n,n}$ is decomposable into commuting Hamilton cycles. So the number of these cycles is $n/2$. Label the vertices of two parts of $K_{n,n}$ by $X = \{1, 3, \dots, 2n-1\}$ and $Y = \{2, 4, \dots, 2n\}$. With no loss of generality, we may assume that $C = G(P_\sigma + P_\sigma^{\mathsf{T}})$ is one of the Hamilton cycles appeared in the commuting decomposition of $K_{n,n}$, where $\sigma = (1\,2\,\dots\,(2n))$. Let $\mathcal{G}$ be the set of all Hamilton cycles in K_{2n} which have no common edge with C and commute with C. By Theorem 4 of [1], any element of $\mathcal{G}$ has the form $G(P_{\sigma^k} + P_{\sigma^k}^{\mathsf{T}})$, where $k \in \{1, \dots, 2n\}$ and k and $2n$ are coprime. For any k, we have $P_{\sigma^k}^{\mathsf{T}} = P_{\sigma^{2n-k}}$, so $|\mathcal{G}| \leqslant \phi(2n)/2$, where ϕ is Euler function. This implies that $\frac{n}{2} \leqslant \frac{\phi(2n)}{2}$. Let $n = 2^t n_o$, for some $t \geqslant 1$ and odd number n_o. Since ϕ is an arithmetic function, $\phi(2n) = 2^t\phi(n_o)$ and thus the above inequality yields that $n_o \leqslant \phi(n_o)$. From this we conclude that $n_o = 1$, so n is a power of 2, as desired. For the other direction, suppose that n is a power of 2. Clearly,

$$\mathcal{H} = \left\{ G\left(P_{\sigma^k} + P_{\sigma^k}^{\mathsf{T}}\right) \middle| \, k \text{ is odd and } k < n \right\}$$

is a set of commuting Hamilton cycles in K_{2n}. For any k, we have $P_{\sigma^k} + P_{\sigma^k}^{\mathsf{T}} = \sum_{i=1}^{2n}(E_{ii'} + E_{i'i})$, where $i' = i + k \pmod{2n}$. Then one can easily see that no pair of $\mathcal{H}$ has common edge. Furthermore, $i \neq i' \pmod 2$ for any i, hence for each $H \in \mathcal{H}$, every edge of H meets X and Y. Since $|\mathcal{H}| = n/2$, $\mathcal{H}$ is a decomposition of $K_{n,n}$ into commuting Hamilton cycles. $\square$

3. Commuting graphical matrices and Hadamard matrices

In Proposition 2.3.6 of [3], Heinze has determined the centralizers of the complete graph K_n and the complete bipartite graph $K_{n,n}$. In this section, we obtain the centralizers of some other graphs. First we find the maximum size of a linearly independent set of commuting graphical matrices.

Theorem 5. *If $\mathcal{A}$ is a linearly independent set of commuting graphical matrices in $M_n(\mathbb{R})$, then $|\mathcal{A}| \leqslant n-1$. Moreover, if the equality holds, then there exists a Hadamard matrix of order n. In particular, in the equality case, either $n \leqslant 2$ or n is divisible by 4.*

Proof. Since $\mathcal{A}$ is a commuting diagonalizable set, there exists an orthogonal matrix $P = [p_{ij}] \in M_n(\mathbb{R})$ such that $P^{\mathsf{T}}\mathcal{A}P$ is a linearly independent set of diagonal matrices with trace 0. This shows that $|\mathcal{A}| \leqslant n-1$. Now, suppose that $|\mathcal{A}| = n-1$ and let $\mathcal{A} = \{A_1, \dots, A_{n-1}\}$. Since $\operatorname{tr} A_j = 0$, it is easy to see that $\{I_n\} \cup P^{\mathsf{T}}\mathcal{A}\,P$ is a linearly independent set of diagonal matrices in $M_n(\mathbb{R})$. So it is a basis for the linear space of diagonal matrices. Therefore, for $j = 1, \dots, n$, there exist real numbers $\lambda_{j0}, \dots, \lambda_{j,n-1}$ such that

$$E_{jj} = \lambda_{j0}I_n + \lambda_{j1}P^{\mathsf{T}}A_1P + \cdots + \lambda_{j,n-1}P^{\mathsf{T}}A_{n-1}P,$$

and so

$$PE_{jj}P^{\mathsf{T}} = \lambda_{j0}I_n + \lambda_{j1}A_1 + \cdots + \lambda_{j,n-1}A_{n-1}. \tag{2}$$

Since $\operatorname{tr} A_1 = \cdots = \operatorname{tr} A_{n-1} = 0$, the relation (2) implies that $1 = \operatorname{tr}(PE_{jj}P^{\mathsf{T}}) = n\lambda_{j0}$, for $j = 1, \dots, n$. Hence $\lambda_{j0} = 1/n$, for $j = 1, \dots, n$. Furthermore, for any i, j, the (i, i)-entry of $PE_{jj}P^{\mathsf{T}}$ is p_{ij}^2. On the other hand, all diagonal entries of the right side of the equality (2) are λ_{j0}, thus for any i, j, $p_{ij}^2 = 1/n$. Now, if we put $H = n^{\frac{1}{2}}P$, then H is a $(-1, 1)$-matrix and $HH^{\mathsf{T}} = nI_n$. Therefore H is a Hadamard matrix of order n and by Theorem 18.1 of [5], either $n \leqslant 2$ or n is divisible by 4. $\square$

Remark 6. Let n be a power of 2. By Theorem 1 of [1], the complete graph K_n is decomposable into commuting perfect matchings. It is clear that the adjacency matrices of these perfect matchings are linearly independent. Hence there exists a linearly independent set of commuting graphical matrices of size $n-1$, when n is a power of 2.

4. The centralizers of some families of graphs

For any $n \geqslant 1$, let $\mathcal{P}_n(x)$ be the characteristic polynomial of the path P_n. In [2], Beezer has asked when a polynomial in a graphical matrix A yields a graphical matrix. In Theorem 3.1 of [2], a solution in the case that $G(A)$ is a path has been given as follows.

Theorem A. *For any $n \geqslant 1$, let A_n be the adjacency matrix of P_n and suppose that $f(x) \in \mathbb{R}[x]$ is a non-zero polynomial of degree less than n. Then $f(A_n)$ is a graphical matrix if and only if $f(x) = \mathcal{P}_{2i-1}(x)$, for some i, $1 \leqslant i \leqslant \lfloor n/2 \rfloor$.*

Theorem 7. *For any $n \geqslant 1$, let A_n be the adjacency matrix of P_n. Then*

$$\mathcal{C}(P_n) = \{\mathcal{P}_{2i-1}(A_n) | 1 \leqslant i \leqslant \lfloor n/2 \rfloor\}.$$

Proof. We know that all eigenvalues of P_n are mutually distinct. So the characteristic polynomial and the minimal polynomial of A_n are the same. Thus by Theorem 39.1.2 of [4], any matrix commuting with A_n is a polynomial in A_n. Now, by Theorem A the assertion is proved. $\square$

For any $n, m \geqslant 1$, let $W_{n,m}$ be the graph obtained from K_n by joining m new vertices to each vertex of K_n as pendant vertices. Note that the graph $W_{n,m}$ has $n(m+1)$ vertices.

Theorem 8. *For any $n, m \geqslant 1$, the only connected graph commuting with $W_{n,m}$ is $W_{n,m}$. In addition, the graph $K_{1,m}$ is self-centralizer.*

Proof. We label the vertices of $W_{n,m}$ by $1, \ldots, n$ for the vertices of K_n and for any i, $1 \leqslant i \leqslant n$, we use the labels $n + (i-1)m + 1, \ldots, n + im$ for the vertices that are adjacent to the ith vertex of K_n. Thus the adjacency matrix of $W_{n,m}$ is

$$A = \begin{bmatrix} J_n - I_n & I_n \otimes \mathbf{j}_m \\ I_n \otimes \mathbf{j}_m^{\mathsf{T}} & 0_{nm} \end{bmatrix}.$$

Suppose that

$$B = \begin{bmatrix} X & Y \\ Y^{\mathsf{T}} & Z \end{bmatrix}$$

commutes with A, where $X = [x_{ij}] \in M_n(\mathbb{R})$. The equality $AB = BA$ yields that $(I_n \otimes \mathbf{j}_m^{\mathsf{T}})Y = Y^{\mathsf{T}}(I_n \otimes \mathbf{j}_m)$. It is not hard to see that the latter equality implies that there exists a symmetric $(0, 1)$-matrix $C = [c_{ij}] \in M_n(\mathbb{R})$ such that $Y = C \otimes \mathbf{j}_m^{\mathsf{T}}$. Moreover, it is easily verified that for every two matrices $R \in M_{r\times k}(\mathbb{R})$ and $S \in M_{k\times s}(\mathbb{R})$, the equality $R(S \otimes \mathbf{j}_m) = RS \otimes \mathbf{j}_m$ holds. Now, the relation $AB = BA$ yields that $(J_n - I_n)Y + (I_n \otimes \mathbf{j}_m)Z = X(I_n \otimes \mathbf{j}_m)$ and hence $(I_n \otimes \mathbf{j}_m)Z = (X - (J_n - I_n)C) \otimes \mathbf{j}_m$. Let $Z = [P_{ij}]_{1\leqslant i,j\leqslant n}$, where P_{ij} is a $(0, 1)$-matrix in $M_m(\mathbb{R})$. By multiplying the latter equality by $I_n \otimes \mathbf{j}_m^{\mathsf{T}}$ from the right side, we have

$$[\sigma(P_{ij})] = m\,(X - (J_n - I_n)C)\,, \tag{3}$$

where $\sigma(P_{ij})$ is the sum of all entries of the matrix P_{ij}. For any i, we have $\sigma(P_{ii}) = m(x_{ii} - \sum_{j\neq i} c_{ij})$. Since Z is a $(0, 1)$-matrix and all diagonal entries of X are 0, from the latter equality we conclude that C is a diagonal matrix. Note that Z is a symmetric matrix and $\sigma(P_{ij}) = \mathbf{j}_m P_{ij} \mathbf{j}_m^{\mathsf{T}}$, for any i, j. Hence by (3), we deduce that $(J_n - I_n)C$ is a symmetric matrix. This yields that $c_{11} = \cdots = c_{nn}$. Obviously, if $C = 0$, then $G(B)$ is not connected. So assume that $C = I_n$. Since $\sigma(P_{ij}) \geqslant 0$ and X is a $(0, 1)$-matrix, (3) implies that $X = J_n - I_n$ and thus $Z = 0$. Therefore we obtain that $A = B$, as desired. Furthermore, if $n = 1$ and $C = 0$, then from (3) we conclude that $B = 0$. This shows that $W_{1,m} = K_{1,m}$ is self-centralizer. $\square$

In the following, for any $n \geqslant 1$, F_n denotes the *friendship graph* on $2n + 1$ vertices, that is, the graph formed by n triangles which have a common vertex.

Theorem 9. *For any $n \geqslant 1$, the graph F_n is self-centralizer.*

Proof. Label the vertices of F_n such that, for $i = 1, \ldots, n$, $\{1, 2i, 2i + 1\}$ be the vertices of the ith triangle of F_n. Let X be a non-zero graphical matrix such that $G(X)$ commutes with F_n. Suppose that Y is the adjacency matrix of F_n and

$$X = \begin{bmatrix} 0 & x \\ x^{\mathsf{T}} & A \end{bmatrix},$$

where x is a $(0, 1)$-vector in $M_{1\times 2n}(\mathbb{R})$. Also, we have

$$Y = \begin{bmatrix} 0 & \mathbf{j}_{2n} \\ \mathbf{j}_{2n}^{\mathsf{T}} & B \end{bmatrix}, \quad \text{where } B = I_n \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in M_{2n}(\mathbb{R}).$$

Using the equality $XY = YX$, we find that $\mathbf{j}_{2n}A = xB$ and $\mathbf{j}_{2n}^{\mathsf{T}}x + BA = x^{\mathsf{T}}\mathbf{j}_{2n} + AB$. By multiplying the latter equality by $\mathbf{j}_{2n}$, we obtain that $2nx + \mathbf{j}_{2n}BA = s\mathbf{j}_{2n} + \mathbf{j}_{2n}AB$, where s is the sum of all entries of x. Using the equations $\mathbf{j}_{2n}A = xB$, $\mathbf{j}_{2n}B = \mathbf{j}_{2n}$, and $B^2 = I_n$, we have

$$2nx + xB = s\mathbf{j}_{2n} + x. \tag{4}$$

If $x = 0$, then the relation $\mathbf{j}_{2n}A = xB$ yields that $A = 0$. This is impossible, since $X \neq 0$. Thus we may assume that for some index i, the ith entry of x is 1. Moreover, for $n = 1$ the assertion is clear, so assume that $n \geqslant 2$. To obtain a contradiction, suppose that $s \leqslant 2n - 1$. The ith entry of left side of (4) is at least $2n$ and the ith entry of its right side is $s + 1$. This implies that $s \geqslant 2n - 1$ and so exactly one entry of x is 0. Hence there exists an index k such that the kth entry of both x and xB are 1. Now, the kth entry of the left side and the right side of (4) are $2n + 1$ and $2n$, respectively. This is impossible, so $s = 2n$. This means that $x = \mathbf{j}_{2n}$. Therefore $\mathbf{j}_{2n}A = \mathbf{j}_{2n}B = \mathbf{j}_{2n}$, hence $G(A)$ is a perfect matching with $2n$ vertices. Thus $G(X) \simeq F_n$ and the proof is complete. □

For any $n \geqslant 2$ and $m \geqslant 3$, let $\mathcal{S}_{n,m}$ be the *starlike tree* which obtained by joining a vertex υ to m end points of m mutually disjoint paths P_n. If we label the vertex υ by 1 and all vertices with distance i of υ by $\{(i-1)m+2, \ldots, im+1\}$ for $i = 1, \ldots, n$, then the adjacency matrix of $\mathcal{S}_{n,m}$ is

$$\mathcal{A}_{n,m} = \left[\begin{array}{c|cccc} 0 & \mathbf{j}_m & 0 & \cdots & 0 \\ \hline \mathbf{j}_m^{\mathsf{T}} & 0 & I_m & & \bigcirc \\ 0 & I_m & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & I_m \\ 0 & \bigcirc & & I_m & 0 \end{array}\right].$$

Theorem 10. *Let $n \geqslant 2$ and $m \geqslant 3$. Then every non-zero element of $\mathcal{C}(\mathcal{S}_{n,m})$ has the form*

$$\left[\begin{array}{c|cccc} 0 & \mathbf{j}_m & 0 & \cdots & 0 \\ \hline \mathbf{j}_m^{\mathsf{T}} & 0 & P & & \bigcirc \\ 0 & P & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & P \\ 0 & \bigcirc & & P & 0 \end{array}\right],$$

where $G(P)$ is a perfect matching on m vertices. In particular, the graph $\mathcal{S}_{n,m}$ is self-centralizer.

Proof. Let

$$X = \left[\begin{array}{c|ccc} 0 & x_1 & \cdots & x_n \\ \hline x_1^{\mathsf{T}} & X_{11} & \cdots & X_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\mathsf{T}} & X_{n1} & \cdots & X_{nn} \end{array}\right]$$

be a non-zero element of $\mathcal{C}(\mathcal{S}_{n,m})$. Since the first rows of two matrices $\mathcal{A}_{n,m}X$ and $X\mathcal{A}_{n,m}$ are the same, we conclude that

$$\mathbf{j}_m X_{11} = x_2, \tag{5}$$

$$\mathbf{j}_m X_{1j} = x_{j-1} + x_{j+1}, \quad \text{for } j = 2, \ldots, n-1, \tag{6}$$

and

$$\mathbf{j}_m X_{1n} = x_{n-1}. \tag{7}$$

Also, by considering the (i, j)-blocks of two sides of the equality $\mathcal{A}_{n,m}X = X\mathcal{A}_{n,m}$ for $i, j \in \{2, \ldots, n+1\}$, we obtain the following equations.

$$\mathbf{j}_m^{\mathsf{T}} x_1 + X_{21} = x_1^{\mathsf{T}}\mathbf{j}_m + X_{12}. \tag{8}$$

$$\mathbf{j}_m^{\mathsf{T}} x_j + X_{2j} = X_{1,j-1} + X_{1,j+1}, \quad \text{for } j = 2, \ldots, n-1. \tag{9}$$

$$\mathbf{j}_m^{\mathsf{T}} x_n + X_{2n} = X_{1,n-1}. \tag{10}$$

$$X_{i-1,j} + X_{i+1,j} = X_{i,j-1} + X_{i,j+1}, \quad \text{for any } i, j,\ 2 \leqslant i, j \leqslant n-1. \tag{11}$$

$$X_{i-1,n} + X_{i+1,n} = X_{i,n-1}, \quad \text{for } i = 2, \ldots, n-1. \tag{12}$$

$$X_{n-1,n} = X_{n,n-1}. \tag{13}$$

For $k = 1, \ldots, n$, define $\mathcal{D}_k = \{X_{s,n-k+s} \mid 1 \leqslant s \leqslant k\}$ and let $\mathcal{D}_0 = \{0_m\}$. By induction on k, we prove the following assertions for $k = 1, \ldots, n-1$.

$$\mathcal{P}_k: \quad x_{n-k+1} = 0, \mathcal{D}_{k-1} = \{0\}, \quad \text{and} \quad |\mathcal{D}_{k+1}| = 1.$$

First assume that $k = 1$. By multiplying two sides of (10) by $\mathbf{j}_m$, we find that $mx_n + \mathbf{j}_m X_{2n} = \mathbf{j}_m X_{1,n-1}$. Using the Eq. (6) for $j = n-1$, we obtain that $(m-1)x_n + \mathbf{j}_m X_{2n} = x_{n-2}$. Since $m \geqslant 3$ and x_{n-2} is a $(0, 1)$-vector, $x_n = 0$ and so by (10), $\mathcal{P}_1$ is proved. Now, suppose that $k \geqslant 2$ and $\mathcal{P}_t$ is true for any t, $1 \leqslant t \leqslant k-1$. If $k \geqslant 3$, then we first show that

$$X_{2,n-k+1} = X_{3,n-k+2} = \cdots = X_{k,n-1}. \tag{14}$$

By (11), we have $X_{s-1,n-k+s} + X_{s+1,n-k+s} = X_{s,n-k+s-1} + X_{s,n-k+s+1}$, for $s = 2, \ldots, k-1$. Since $|\mathcal{D}_{k-1}| = 1$, $X_{s-1,n-k+s} = X_{s,n-k+s+1}$. Thus $X_{s,n-k+s-1} = X_{s+1,n-k+s}$, for any s, $2 \leqslant s \leqslant k-1$. This establishes (14).

Now, we show that $x_{n-k+1} = 0$. If $k \geqslant 3$, then since $\mathcal{P}_{k-2}$ is true, $x_{n-k+3} = 0$. By multiplying two sides of (9) by $\mathbf{j}_m$ for $j = n-k+1$ and using the relations (6) and (7), we find that $mx_{n-k+1} + \mathbf{j}_m X_{2,n-k+1} = \mathbf{j}_m X_{1,n-k} + \mathbf{j}_m X_{1,n-k+2} = x_{n-k-1} + x_{n-k+1} + x_{n-k+1}$, for any $k \geqslant 2$. By applying (14), we obtain that $(m-2)x_{n-k+1} + \mathbf{j}_m X_{k,n-1} = x_{n-k-1}$. Using the Eq. (12) for $i = k$, we have

$$X_{k,n-1} = X_{k-1,n} + X_{k+1,n}. \tag{15}$$

By $|\mathcal{D}_{k-1}| = 1$, we have $X_{k-1,n} = X_{1,n-k+2}$. Moreover, since $x_{n-k+3} = 0$, the Eq. (6) for $j = n-k+2$ implies that $\mathbf{j}_m X_{1,n-k+2} = x_{n-k+1}$. Now, by multiplying two sides of (15) by $\mathbf{j}_m$, we obtain that $\mathbf{j}_m X_{k,n-1} = x_{n-k+1} + \mathbf{j}_m X_{k+1,n}$. This yields that $(m-1)x_{n-k+1} + \mathbf{j}_m X_{k+1,n} = x_{n-k-1}$. Since $m \geqslant 3$ and x_{n-k-1} is a $(0, 1)$-vector, the latter equation implies that $x_{n-k+1} = 0$. This yields that $X_{1,n-k+2} = 0$. From $|\mathcal{D}_{k-1}| = 1$ and $X_{1,n-k+2} \in \mathcal{D}_{k-1}$, we conclude that $\mathcal{D}_{k-1} = \{0\}$. Thus the Eq. (9) for $j = n-k+1$ implies that $X_{2,n-k+1} = X_{1,n-k}$. On the other hand, $\mathcal{D}_{k-1} = \{0\}$ yields that $X_{k-1,n} = 0$. So the Eq. (12) for $i = k$ implies that $X_{k,n-1} = X_{k+1,n}$. Now, by applying (14), we obtain that $|\mathcal{D}_{k+1}| = 1$ and so $\mathcal{P}_k$ is true. Therefore $\mathcal{P}_t$ holds for $t = 1, \ldots, n-1$.

We have $x_2 = 0$, so by (5) we find that $X_{11} = 0$ and thus $D_n = \{0\}$. Since $\{X_{12}, X_{n-1,n}\} \subseteq D_{n-1}$, we conclude that $X_{12} = X_{n-1,n}$. Moreover, we know that $X_{n,n-1} = X_{n-1,n}^{\mathsf{T}}$, therefore (13) yields that X_{12} is a symmetric matrix. Hence (8) implies that the all entries of x_1 are the same. On the other hand, using the Eq. (6) for $j = 2$, we find that $\mathbf{j}_m X_{12} = x_1 + x_3$. Since X is a non-zero matrix, $X_{12} \neq 0$. Thus the equality $x_3 = 0$ yields that $x_1 = \mathbf{j}_m$ and so every column of X_{12} has exactly one non-zero entry. But X_{12} is a symmetric matrix, hence it is a permutation matrix. Now, clearly $G(X_{12})$ is a perfect matching with m vertices and $\mathcal{S}_{n,m} \simeq G(X)$, as desired. □

We recall that the *Cartesian product* $G \times H$ of two graphs G and H is the graph whose vertex set is the cartesian product of the vertex sets of G and H such that two vertices (g_1, h_1) and (g_2, h_2) are adjacent if (i)$g_1 = g_2$ and $h_1 - h_2$ is an edge of H or (ii)$h_1 = h_2$ and $g_1 - g_2$ is an edge of G. The following theorem shows that the commutativity of graphs is invariant under Cartesian product.

Theorem 11. *Let $\{G_1, G_2\}$ and $\{H_1, H_2\}$ be two pairs of graphs with the same order. Then $\{G_1 \times H_1, G_2 \times H_2\}$ is a pair of commuting graphs if and only if $\{G_1, G_2\}$ and $\{H_1, H_2\}$ are commuting pairs.*

Proof. Let $A_i \in M_n(\mathbb{R})$ and $B_i \in M_m(\mathbb{R})$ be the adjacency matrices of G_i and H_i, respectively, for $i = 1, 2$. Then the adjacency matrix of the graph $G_i \times H_i$ is $A_i \otimes I_m + I_n \otimes B_i$, for $i = 1, 2$. It is straightforward to see that two matrices $A_1 \otimes I_m + I_n \otimes B_1$ and $A_2 \otimes I_m + I_n \otimes B_2$ commute if and only if $(A_1A_2 - A_2A_1) \otimes I_m + I_n \otimes (B_1B_2 - B_2B_1) = 0$. Clearly, this equation is equivalent to $A_1A_2 - A_2A_1 = B_1B_2 - B_2B_1 = 0$. □

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